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Symmetry-breaking boundary states for WZW models

Daniel Blakeley and **Andreas Recknagel**

King's College London, Department of Mathematics

Strand, London WC2R 2LS, UK

daniel.blakeley@kcl.ac.uk, andreas.recknagel@kcl.ac.uk

Abstract

Starting with the $\mathrm{SU}(2)_k$ WZW model, we construct boundary states that generically preserve only a parafermion times Virasoro subalgebra of the full affine Lie algebra symmetry of the bulk model. The boundary states come in families: intervals for generic k, quotients of SU(2) by discrete groups if k is a square. In that case, special members of the families can be viewed as superpositions of rotated Cardy branes. Using embeddings of SU(2) into higher groups, the new boundary states can be lifted to symmetry-breaking branes for other WZW models.

1. Introduction

D-branes have become an extremely important ingredient of string theory. Since their discovery [1], it has been clear that they have a world-sheet description as conformal boundary conditions, or boundary states. The CFT approach is distinguished from the target space picture of branes in that it does not refer to classical geometry – which makes it harder to interpret its results, but broadens the scope towards D-branes that would be hard to find based on classical intuitions.

The world-sheet construction and classification of D-branes is rather well under control if one restricts to boundary conditions that preserve the maximal symmetry, in rational conformal field theories [2,3]. If the CFT in question has a sigma model interpretation, one can often relate the CFT boundary states to D-brane submanifolds in the target; in WZW models, e.g., maximally symmetric boundary states correspond to conjugacy classes (perhaps rotated or twisted) in the group target [4]. For other rational backgrounds like Gepner models [5], the relation may already be more intricate due to lines of marginal stability in the bulk moduli space [6].

Consistency of string models on world-sheets with boundary only requires branes to preserve conformal symmetry, and therefore it is natural to study symmetry-breaking boundary states. However, there are at present no general principles to make up for the loss of symmetry, thus it is very difficult to construct symmetry-breaking boundary conditions. Apart from Virasoro minimal models, complete lists of conformal boundary states are known only for $c = 1$ theories [7,8,9,10] – see also the earlier works [11,12,13] – and (most probably) for the Liouville theory [14,15,16,17,18] (where it is more difficult to decide completeness due to the non-compactness of the model).

Partially symmetry-breaking boundary state for WZW and coset models were studied in particular in [19,20]. The maximal symmetry algebra is broken up into a subalgebra and its commutant, so that one can in particular choose twisted gluing conditions for the subalgebra (and standard gluing conditions for the commutant).

Finding a geometric interpretation of symmetry-breaking branes is usually difficult, though [19] provides a target picture for special cases; see also [21]. On the other hand, one expects boundary conditions with reduced symmetry to appear naturally in connection with boundary renormalisation group flows, such as tachyon condensations. For example, it turns out that Sen's process of dimensional transmutation [22] can be described using conformal free boson boundary states [13,9]. We hope that the families of symmetry-breaking SU(2) boundary states constructed in this article will find similar applications to string theory (e.g. in relation to branes in the NS 5-brane background [23]) or in condensed matter physics (where SU(2) boundary states were used to solve the Kondo effect [24]).

Apart from abstract results like the g-theorem [25], too little is known at present about the details of RG flows to turn their study into an efficient method to construct symmetry-breaking boundary states. Instead, we will exploit and extend the methods from [19,20] here.

The paper is organised as follows: As the construction of our boundary states rests on a decomposition of the $\mathrm{SU}(2)_k$ state space into parafermions and free bosons, we start by reviewing this in some detail. In Section 3, we write down an ansatz for symmetry-breaking boundary states and show that it satisfies Cardy's condition, i.e. that the overlaps of two such boundary states can be regarded as an open string partition function. The analysis is performed for the case that the level k of the WZW model is a square, while Section 4 deals with the general case, where a smaller family of boundary states results. In Section 5, we show that, for special values of the parameters, the new boundary states can be viewed as intersecting configurations of maximally symmetric $\mathrm{SU}(2)_k$ boundary states; it appears that the new families of boundary states interpolate between branes of different dimension in target space. Using embeddings of SU(2), we generate symmetry-breaking boundary states for higher rank WZW targets G in Section 6, before concluding with a list of open problems.

2. Decomposition of representations and Ishibashi states

The idea of our construction is very simple: start from the bulk Hilbert space of the SU(2) WZW model at level k (with diagonal modular invariant partition function), decompose each $\mathrm{SU}(2)_k$ irrep into (sums of) products of parafermion times U(1) irreps, then decompose the latter further into Virasoro irreps – both in the left- and the right-moving sector. Suitable left-right combinations then provide Ishibashi states preserving the reduced parafermion times Virasoro symmetry. In the next section, we will propose symmetry-breaking boundary states as linear combinations of those Ishibashi states and show that Cardy's conditions are satisfied. Here, we review the decompositions of representations which provide the Ishibashi states.

We work with an SU(2) WZW model with diagonal modular invariant bulk partition function; the bulk state space is

$$H = \bigoplus_{J=0,\dots,\frac{k}{2}} H_J^{\mathrm{SU}(2)_k} \otimes \bar{H}_J^{\mathrm{SU}(2)_k} \quad .$$

The irreducible $\mathrm{SU}(2)_k$ representations $H_J^{\mathrm{SU}(2)_k}$ can be decomposed with respect to the smaller parafermion times free boson symmetry algebra $\mathrm{SU}(2)_k/\mathrm{U}(1)_k \otimes U(1)_k$, where

more precisely $\mathrm{U}(1)_k$ denotes the abelian current algebra at radius $r = \sqrt{k}\, r_{s.d.}$, extended by the local fields $\exp(\pm i\sqrt{2k}\, X(z))$ – using the conventions of [19]. The chiral algebra $\mathcal{A}$ of the parafermion theory has irreducible representations labelled by (J, n), where $J \in \frac{1}{2}\mathbb{Z}$ with $0 \leq J \leq \frac{k}{2}$ and where $n = -k+1, \ldots, k$ is integer such that $2J + n$ is even; there is a field identification

$$(\tfrac{k}{2} - J, n + k) \sim (J, n) \ . \tag{2.1}$$

The chiral $\mathrm{SU}(2)_k$ modules decompose as

$$H_J^{\mathrm{SU}(2)_k} = \bigoplus_{\substack{n \\ 2J+n \text{ even}}} H^{\mathrm{PF}}_{(J,n)} \otimes H_n^{\mathrm{U}(1)_k}$$

which implies, at the level of characters, that

$$\chi_J^{\mathrm{SU}(2)_k}(q, z) = \sum_{\substack{n=-k+1,\ldots,k \\ 2J+n \text{ even}}} \chi^{\mathrm{PF}}_{(J,n)}(q)\, \chi_n^{\mathrm{U}(1)_k}(q, z) \ . \tag{2.2}$$

We will not need the explicit form of the parafermion characters $\chi^{PF}_{(J,n)}(q)$, which is for example given [26,27,19], but the symmetries

$$\chi^{PF}_{(J,n)}(q) \equiv \chi^{PF}_{(\frac{k}{2}-J, n+k)}(q) = \chi^{PF}_{(J,-n)}(q) \tag{2.3}$$

will be important later on.
The $U(1)_k$ characters are given by

$$\chi_n^{U(1)_k}(q, z) = \frac{\Theta_{n,k}(q, z)}{\eta(q)} = \frac{1}{\eta(q)} \sum_{m \in \mathbb{Z}} q^{k(m+\frac{n}{2k})^2} e^{2\pi i z k (l + \frac{n}{2k})} \tag{2.4}$$

where $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ is the Dedekind eta function. In particular, we have a decomposition

$$\chi_n^{U(1)_k}(q) = \frac{1}{\eta(q)} \sum_{m \in \mathbb{Z}} q^{k(m+\frac{n}{2k})^2} = \sum_{m \in \mathbb{Z}} \chi^{U(1)}_{m,n}(q) \tag{2.5}$$

into characters of irreducible U(1) representations $H^{\mathrm{U}(1)}_{m,n}$ built up over ground states of conformal dimension

$$h_{m,n} = k\Big(m + \frac{n}{2k}\Big)^2 \ . \tag{2.6}$$

To be able to construct boundary states which break the symmetry down to PF $\times$ Vir, we need to further decompose the U(1) representations into Virasoro representations: At central charge $c = 1$, every Virasoro Verma module is irreducible except if the highest weight is the square of a half-integer; therefore only irreducible U(1)-representation with

highest weight $h_{(m,n)} = j^2$ for some $j \in \frac{1}{2}\mathbb{Z}$ are reducible with respect to the Virasoro algebra. Comparing to (2.6), we see that for generic k this can only occur in the vacuum sector $m = n = 0$; alternatively we need that the level k of the $\mathrm{SU}(2)_k$ model is a square

$$k = \kappa^2 \quad \text{for some} \quad \kappa \in \mathbb{Z}_+ \tag{2.7}$$

and that the $\mathrm{U}(1)_k$ representation label is

$$n = \nu \cdot \kappa \quad \text{for some} \quad \nu = -\kappa + 1, \ldots, \kappa \ . \tag{2.8}$$

We will address the case of generic k in Section 4 below, but for the time being we assume that the two conditions (2.7) and (2.8) above are satisfied. In particular, we will restrict ourselves to Virasoro Ishibashi states associated with such degenerate representations of the Virasoro algebra in building up our symmetry-breaking boundary states.

An irreducible U(1) representation with lowest conformal dimension $h_{m,n} = j^2$ for some $j \in \frac{1}{2}\mathbb{Z}$ – i.e. coming from a $\mathrm{U}(1)_k$ module with label as in (2.8) –, decomposes as

$$\mathcal{H}^{U(1)}_{m,n=\nu\kappa} = \bigoplus_{l=0}^{\infty} \mathcal{H}^{\mathrm{Vir}}_{|m\kappa + \frac{\nu}{2}|+l} \ , \tag{2.9}$$

where $\mathcal{H}^{\mathrm{Vir}}_p$ (with $p \geq 0$) denotes the irreducible Virasoro representation with highest weight $h = p^2$. The Virasoro characters for $c = 1$ are given by

$$\begin{aligned} h \neq j^2 \qquad & \chi^{\mathrm{Vir}}_h(q) = \frac{q^h}{\eta(q)} \equiv \vartheta_{\sqrt{2h}}(q) \ , \\ h = j^2 \qquad & \chi^{\mathrm{Vir}}_h(q) = \vartheta_{\sqrt{2}j}(q) - \vartheta_{\sqrt{2}(j+1)}(q) \ . \end{aligned} \tag{2.10}$$

Combining left and right movers, the full $\mathrm{SU}(2)_k$ state space is given by

$$\bigoplus_{J=0,\ldots,\frac{k}{2}} H^{\mathrm{SU}(2)_k}_J \otimes \bar{H}^{\mathrm{SU}(2)_k}_J = \bigoplus_{\substack{J=0,\ldots,\frac{k}{2} \\ n_L,n_R=-k+1,\ldots,k \\ 2J+n_l \mathrm{even} \\ 2J+n_r \mathrm{even}}} H^{\mathrm{PF}}_{(J,n_L)} \otimes \bar{H}^{\mathrm{PF}}_{(J,n_R)^+} \otimes H^{\mathrm{U}(1)_k}_{n_L} \otimes \bar{H}^{\mathrm{U}(1)_k}_{n_R{}^+} \tag{2.11}$$

where the subscripts $(J,n)^+ := (J,-n)$ of parafermion and $n^+ := -n$ of $\mathrm{U}(1)_k$ representations denote conjugate sectors.

Restricting to $k = \kappa^2$, we can apply the above decompositions of left- and right-moving

U(1) representations and obtain an explicit expression for that subspace of the WZW bulk space from which we will build up our symmetry-breaking boundary states:

$$\bigoplus_{J=0,\ldots,\frac{k}{2}} H_J^{SU(2)_k} \otimes \bar{H}_J^{SU(2)_k} \supset \bigoplus_{\substack{J=0,\ldots,\frac{k}{2} \\ \nu_L,\nu_R=\kappa+1,\ldots,\kappa \\ m_L,m_R\in\mathbb{Z} \\ \text{s.t.}\,2J+\nu_L\,\text{even} \\ \text{s.t.}\,2J+\nu_R\,\text{even} \\ l_L,l_R\in\mathbb{Z}_+}} H^{\text{PF}}_{(J,\nu_L\kappa)} \otimes \bar{H}^{\text{PF}}_{(J,-\nu_R\kappa)} \otimes H^{\text{Vir}}_{|\, m_L\kappa+\frac{\nu_L}{2}|+l_L} \otimes \bar{H}^{\text{Vir}}_{|\, m_R\kappa-\frac{\nu_R}{2}|+l_R} . \tag{2.12}$$

We have ignored all contributions from $\mathrm{U}(1)_k$ modules which yield non-degenerate Virasoro representations. Note that this choice also influences what PF representations are contained in the subspace given in (2.12), since the $\mathrm{U}(1)_k$ label is coupled to the PF label. To be able to form PF $\times$ Vir Ishibashi states, we need $(J,\nu_L\kappa) \sim (J,-\nu_R\kappa)$ in the PF part (as Ishibashi states couple a representation on the left to its conjugate on the right) and $h_L = h_R$ in the Virasoro part, i.e. that $|m_L\kappa + \frac{\nu}{2}| + l_L = |m_R\kappa - \frac{\nu}{2}| + l_R$. Switching notations to $r := m_L\kappa + \frac{\nu}{2}$ and $s := -m_R\kappa + \frac{\nu}{2}$, we see that the SU(2) bulk space provides a Virasoro Ishibashi state over highest weight j^2 with $j \in \frac{1}{2}\mathbb{Z}$ whenever $-j \le r, s \le j$ and

$$r + s = \kappa\rho + \nu \ , \quad r - s = \kappa\rho' \quad \text{for some} \ \ \rho,\rho' \in \mathbb{Z} \quad \text{with} \ \ \rho+\rho' \ \text{even}. \tag{2.13}$$

The condition $(J,\nu_L\kappa) \sim (J,-\nu_R\kappa)$ on the parafermion representations simply amounts to $\nu_R = -\nu_L =: \nu$, as long as the level k is odd, or as long as k is even and $J \ne \frac{k}{4}$.

For this last case $J = \frac{k}{4}$, however, a complication arises from the field identification (2.1): Since $(\frac{k}{4}, n_L + k) \sim (\frac{k}{4}, n_L)$, there are additional PF $\times$ Vir Ishibashi states whenever $J = \frac{k}{4}$ and $-n_R = n_L + k$; we call those Ishibashi states $|\frac{k}{4}, \nu\kappa\rangle\rangle_{\text{tw}}$.

3. Boundary state construction for $r = \sqrt{k}\, r_{\text{s.d.}} = \kappa\, r_{\text{s.d.}}$

Boundary states can be written as linear combinations of Ishibashi states, the latter implementing the gluing conditions of the preserved symmetry algebra. For rational models with charge conjugate partition function, one can always form the (maximally symmetric) Cardy boundary states, where the coefficients in the superposition are given in terms of modular S-matrix elements. Those boundary states automatically satisfy Cardy's conditions, requiring that the overlap of two boundary states can be written as an open string partition function [2].

To obtain symmetry-breaking boundary states, one has to deviate from Cardy's construction. We will be guided by the results from [8,9] and [10], where boundary states preserving only conformal symmetry were presented for $c = 1$ models. In particular, [9] studied free

bosons compactified at a radius $r = \frac{M}{N}\, r_{\mathrm{s.d.}}$ where M, N are coprime integers, and found that conformal boundary conditions come in $\mathrm{SU}(2)/(\mathbb{Z}_M \times \mathbb{Z}_N)$ families

$$\| \, g \rangle\!\rangle_{c=1} = \quad 2^{-\frac{1}{4}} \sqrt{MN} \sum_{\substack{j;r,s \\ r-s \equiv 0 \;(\mathrm{mod}\; M) \\ r+s \equiv 0 \;(\mathrm{mod}\; N)}} D^j_{r,s}(g)\, | j; r, s \rangle\!\rangle \; ; \tag{3.1}$$

the summation is over $j, r, s \in \frac{1}{2}\mathbb{Z}$ with $j \geq 0$ and $-j \leq r, s \leq j$, and the coefficients [28]

$$D^j_{r,s}(g) = \sum_{l=\max(0,s-r)}^{\min(j-r,j+s)} \frac{[(j+r)!(j-r)!(j+s)!(j-s)!]^{\frac{1}{2}}}{(j-r-l)!\,(j+s-l)!\,l!\,(r-s+l)!} \tag{3.2}$$

$$\times a^{j+s-l} (a^*)^{j-r-l} b^{r-s+l} (-b^*)^l$$

are matrix elements in a spin j representation of SU(2), with $g \in \mathrm{SU}(2)$ taken in the form

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \; . \tag{3.3}$$

It was shown in [8,9] that these boundary states satisfy Cardy's conditions, and that for special values of the parameter g they reduce to superpositions of Neumann or Dirichlet boundary states.

For the SU(2) case we are interested in now, we have obtained, from the SU(2) modules, $\mathrm{PF} \times \mathrm{Vir}$ Ishibashi states, and we can try to combine the $c = 1$ conformal boundary states from above with Cardy boundary states for the parafermionic part. We propose to consider the following boundary states for SU(2) WZW models with diagonal bulk partition function:

$$\| \, g_\alpha; J_\alpha, n_\alpha \rangle\!\rangle = \mathcal{N} \sum_{\substack{\nu=-\kappa+1,\ldots,\kappa \\ J=0,,\ldots,,\frac{k}{2} \\ 2J+\nu\kappa \;\mathrm{even}}} \; \sum_{\substack{j \in \frac{1}{2}\mathbb{Z}_+ \\ r+s=\kappa\rho+\nu \\ r-s=\kappa\rho' \\ \mathrm{s.t.}\;\; \rho+\rho' \mathrm{even}}} B^{\mathrm{PF}\;(J,\nu\kappa)}_{(J_\alpha,n_\alpha)}\, D^j_{r,s}(g_\alpha)\, | J, \nu\kappa \rangle\!\rangle \otimes | j; r, s \rangle\!\rangle \tag{3.4}$$

$\mathcal{N}$ is some normalisation factor, the summation range is dictated by the criteria (2.13) ensuring existence of degenerate Ishibashi states; furthermore, $g_\alpha \in \mathrm{SU}(2)$ and $D^j_{r,s}(g_\alpha)$ are as in (3.2), while the coefficients $B^{\mathrm{PF}\;(J,\nu\kappa)}_{(J_\alpha,n_\alpha)}$ are as in parafermionic Cardy boundary states, i.e.

$$B^{\mathrm{PF}\;(J,\nu\kappa)}_{(J_\alpha,n_\alpha)} = \frac{S^{\mathrm{PF}}_{(J_\alpha,n_\alpha)\,,\,(J,\nu\kappa)}}{\sqrt{S^{\mathrm{PF}}_{(0,0)\,,\,(J,\nu\kappa)}}} \tag{3.5}$$

with the S-matrix

$$S^{\mathrm{PF}}_{(J,n),(J',n')} = \sqrt{\frac{2}{k}}\, e^{\frac{i\pi n n'}{k}} S^{\mathrm{SU}(2)_k}_{J,J'} \quad \text{with} \quad S^{\mathrm{SU}(2)_k}_{J,J'} = \sqrt{\frac{2}{k+2}}\, \sin\frac{(2J+1)(2J'+1)}{k+2}\pi \tag{3.6}$$

from modular transformations of parafermionic characters (with $q = e^{2\pi i\tau},\ \tilde{q} = e^{-2\pi i/\tau}$)

$$\chi^{PF}_{(J,n)}(\tilde{q}) = \sum_{\substack{(J',n') \\ \text{s.t. } 2J'+n' \text{ even}}} S^{PF}_{(J,n),(J',n')}\, \chi^{PF}_{(J',n')}(q)\ .$$

Our main task in the following is to verify whether the boundary states (3.4) satisfy Cardy's condition: We need to compute the overlap of two such boundary states,

$$\mathcal{A}_{\alpha\beta} = \langle\!\langle\, g_\alpha; J_\alpha, n_\alpha \| \,\tilde{q}^{\frac{1}{2}\left(L_0^{SU(2)_k} + \bar{L}_0^{SU(2)_k} - \frac{c}{12}\right)} \,\| \,g_\beta; J_\beta, n_\beta \rangle\!\rangle \tag{3.7}$$

and perform a modular transformation to the open string channel. It will turn out that the result can indeed be written as a positive integer linear combination of parafermion times Virasoro characters, as required. The computation, however, is rather lengthy and involves a rather intricate interplay of field identification, symmetries of structure constants and SU(2) group representations. We will for simplicity restrict to the case that the level k is the square of an odd number κ at first. The case of even κ is discussed at the end of this section, and the case where k is not a square (where we will make use of the constructions in [10]) in section 4.

We start our computation of the boundary state overlap $\mathcal{A}_{\alpha\beta}$ by recalling that Ishibashi states are orthogonal and normalised in the sense that their self-overlap (with the closed string propagator inserted as in (3.7)) produces characters of the associated representations. In the present case this produces products of parafermion and Virasoro characters, since our Ishibashi states have tensor product form. So the overlap is

$$\mathcal{A}_{\alpha\beta} = \sum_{\substack{(J,\nu\kappa) \\ \text{s.t. } 2J+\nu\kappa \text{ even}}} \quad \sum_{\substack{j\in\frac{1}{2}\mathbb{Z}_+ \\ r+s=\kappa\rho+\nu \\ r-s=\kappa\rho' \\ \text{s.t. } \rho+\rho' \text{ even}}} \overline{B}^{\mathrm{PF}\ \alpha}_{(J,\nu\kappa)}\, B^{\mathrm{PF}\ \beta}_{(J,\nu\kappa)}\, \chi^{\mathrm{PF}}_{(J,\nu\kappa)}(\tilde{q})\, \overline{D}^{j}_{r,s}(g_\alpha)\, D^{j}_{r,s}(g_\beta)\, \chi^{\mathrm{Vir}}_{j^2}(\tilde{q})$$

where we have used an abbreviated notation for the parafermion coefficients B^{PF}; the bar denotes complex conjugation.

We rewrite this in terms of parafermionic and Virasoro contributions,

$$\mathcal{A}_{\alpha\beta} = \sum_{\substack{(J,\nu\kappa) \\ \text{s.t. } 2J+\nu\kappa \text{ even}}} \mathcal{A}^{\mathrm{PF}}_{\alpha\beta;\,(J,\nu\kappa)}\, \tilde{\mathcal{A}}^{\mathrm{Vir}}_{\alpha\beta;\,\nu}\ \ ,$$

with

$$\mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)} = \overline{B}^{{\rm PF}\ \alpha}_{(J,\nu\kappa)}\, B^{{\rm PF}\ \beta}_{(J,\nu\kappa)}\, \chi^{\rm PF}_{(J,\nu\kappa)}(\tilde q) \tag{3.8}$$

and

$$\tilde{\mathcal{A}}^{\rm Vir}_{\alpha\beta;\,\nu} = \sum_{\substack{j\in\frac12\mathbb{Z}_+\\ r+s=\kappa\rho+\nu\\ r-s=\kappa\rho'\\ \text{s.t. }\rho+\rho'\text{ even}}} \overline{D}^j_{r,s}(g_\alpha)\, D^j_{r,s}(g_\beta)\, \chi^{\rm Vir}_{j^2}(\tilde q) \quad . \tag{3.9}$$

We will need to perform the entangled summations over parafermionic and Virasoro indices eventually, but first we resolve the constraints on the ρ, ρ' summation: We note that all the parafermionic constituents in the overlap are invariant under the field identification (2.1), therefore $\mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)} = \mathcal{A}^{\rm PF}_{\alpha\beta;\,(\frac{k}{2}-J,\nu\kappa+k)}$.

On the other hand, the condition $\rho+\rho'$ even in (2.13) changes into $\rho+\rho'$ odd under $n\mapsto n+k$, i.e.

$$\tilde{\mathcal{A}}^{\rm Vir}_{\alpha\beta;\,\nu+\kappa} = \sum_{\substack{j\in\frac12\mathbb{Z}_+\\ r+s=\kappa\rho+\nu\\ r-s=\kappa\rho'\\ \text{s.t. }\rho+\rho'\text{ odd}}} \overline{D}^j_{r,s}(g_\alpha)\, D^j_{r,s}(g_\beta)\, \chi^{\rm Vir}_{j^2}(\tilde q) \quad .$$

Thus we can rewrite

$$\begin{aligned}\mathcal{A}_{\alpha\beta} &= \frac12 \sum_{\substack{(J,\nu\kappa)\\ \text{s.t. }2J+\nu\kappa\text{ even}}} \mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)}\, \tilde{\mathcal{A}}^{\rm Vir}_{\alpha\beta;\,\nu} + \frac12 \sum_{\substack{(J,\nu\kappa)\\ \text{s.t. }2J+\nu\kappa\text{ even}}} \mathcal{A}^{\rm PF}_{\alpha\beta;\,(\frac{k}{2}-J,\nu\kappa+k)}\, \tilde{\mathcal{A}}^{\rm Vir}_{\alpha\beta;\,\nu+\kappa}\\ &= \frac12 \sum_{\substack{(J,\nu\kappa)\\ \text{s.t. }2J+\nu\kappa\text{ even}}} \mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)}\, \mathcal{A}^{\rm Vir}_{\alpha\beta;\,\nu}\end{aligned} \tag{3.10}$$

with $\mathcal{A}^{\rm Vir}_{\alpha\beta;\,\nu}$ defined as in (3.9) but *without* any restriction on $\rho+\rho'$.

To proceed, we first exploit the fact that the Virasoro part is independent of the SU(2) spin J in order to perform the J summation in the parafermion contribution – after splitting into integer J (coupled to even $\nu\kappa$ because of the constraint $2J+\nu\kappa\equiv 0 \pmod 2$) and half-odd integer $J\in\mathbb{Z}+\frac12$ (coupled to odd $\nu\kappa$). We perform a modular transformation of the parafermion characters and use the explicit form of the parafermionic coefficients (3.5):

$$\sum_{\substack{J\\ J\in\mathbb{Z}}} \mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)} = \frac{2}{k} \sum_{\substack{J\\ J\in\mathbb{Z}}} \sum_{\substack{(J',n')\\ \text{s.t. }2J'+n'\text{ even}}} e^{\frac{i\pi\nu}{\kappa}(n'+n_\beta-n_\alpha)}\, \frac{S^{{\rm SU(2)}_k}_{J,J_\alpha} S^{{\rm SU(2)}_k}_{J,J_\beta} S^{{\rm SU(2)}_k}_{J,J'}}{S^{{\rm SU(2)}_k}_{0,J}}\, \chi^{\rm PF}_{(J',n')}(q) \ .$$

After inserting $\frac12\big(1+(-1)^{2J}\big)$, which projects onto integer J, and using the identity $(-1)^{2J} S^{{\rm SU(2)}_k}_{J,J'} = S^{{\rm SU(2)}_k}_{J,\frac{k}{2}-J'}$, the Verlinde formula yields

$$\sum_{\substack{J\\ J\in\mathbb{Z}}} \frac{S^{{\rm SU(2)}_k}_{J,J_\alpha} S^{{\rm SU(2)}_k}_{J,J_\beta} S^{{\rm SU(2)}_k}_{J,J'}}{S^{{\rm SU(2)}_k}_{0,J}} = \sum_J \frac{1+(-1)^{2J}}{2}\, \frac{S^{{\rm SU(2)}_k}_{J,J_\alpha} S^{{\rm SU(2)}_k}_{J,J_\beta} S^{{\rm SU(2)}_k}_{J,J'}}{S^{{\rm SU(2)}_k}_{0,J}} = \frac12\Big(N^{J'}_{J_\alpha,J_\beta} + N^{\frac{k}{2}-J'}_{J_\alpha,J_\beta}\Big) \ .$$

With this, we obtain

$$\sum_{\substack{J\\ J\in\mathbb{Z}}} \mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)} = \frac{2}{k} \sum_{\substack{(J',n')\\ \text{s.t. }2J'+n'\text{ even}}} \frac{1}{2}\Big(N^{J'}_{J_\alpha,J_\beta} + N^{\frac{k}{2}-J'}_{J_\alpha,J_\beta}\Big)\, e^{\frac{i\pi\nu}{\kappa}(n'+n_\beta-n_\alpha)}\ \chi^{\rm PF}_{(J',n')}(q)$$
$$= \frac{2}{k} \sum_{\substack{(J',n')\\ \text{s.t. }2J'+n'\text{ even}}} e^{\frac{i\pi\nu}{\kappa}(n'+n_\beta-n_\alpha)}\ N^{J'}_{J_\alpha,J_\beta}\ \chi^{\rm PF}_{(J',n')}(q)\ \ ; \tag{3.11}$$

the simplification in the last step arises after a change of summation variables $(J',n')\mapsto(\frac{k}{2}-J',n'+k)$ in the $N^{\frac{k}{2}-J'}_{J_\alpha,J_\beta}$, which leaves the parafermionic characters invariant and also the exponential because ν is even here for integer J.
In the same way, one can perform the sum over $J\in\mathbb{Z}+\frac{1}{2}$, using the projector $\frac{1}{2}\big(1-(-1)^{2J}\big)$ and recalling that here ν is odd; at the end of the day, one arrives at the same expression as for integer J:

$$\sum_{\substack{J\\ J\in\mathbb{Z}+\frac{1}{2}}} \mathcal{A}^{\rm PF}_{\alpha\beta;\,(J,\nu\kappa)} = \frac{2}{k} \sum_{\substack{(J',n')\\ \text{s.t. }2J'+n'\text{ even}}} \frac{1}{2}\Big(N^{J'}_{J_\alpha,J_\beta} - N^{\frac{k}{2}-J'}_{J_\alpha,J_\beta}\Big)\, e^{\frac{i\pi\nu}{\kappa}(n'+n_\beta-n_\alpha)}\ \chi^{\rm PF}_{(J',n')}(q)$$
$$= \frac{2}{k} \sum_{\substack{(J',n')\\ \text{s.t. }2J'+n'\text{ even}}} e^{\frac{i\pi\nu}{\kappa}(n'+n_\beta-n_\alpha)}\ N^{J'}_{J_\alpha,J_\beta}\ \chi^{\rm PF}_{(J',n')}(q)\ \ . \tag{3.12}$$

To complete the calculation of the overlap between two symmetry-breaking boundary states, we return to the Virasoro contribution to (3.10). We need to evaluate

$$\mathcal{A}^{\rm Vir}_{\alpha\beta;\,\nu} = \sum_{\substack{j;r,s\\ j\in\frac{1}{2}\mathbb{Z}_+,\,-j\le r,s\le j\\ r-s\equiv 0(\mathrm{mod}\kappa)\\ r+s\equiv\nu(\mathrm{mod}\kappa)}} \overline{D}^{j}_{r,s}(g_\alpha)\, D^{j}_{r,s}(g_\beta)\ \chi^{\rm Vir}_{j^2}(\tilde q) \tag{3.13}$$

for $g_\alpha,g_\beta\in\mathrm{SU}(2)$. The restrictions on the permissible r and s can be treated in a similar way to [9] , leading to the insertion of projection operators into the representation matrix D^j. To see this, we first implement the mod κ requirements on r,s with the help of primitive $\kappa^{\rm th}$ roots of unity:

$$\mathcal{A}^{\rm Vir}_{\alpha\beta;\,\nu} = \frac{1}{\kappa^2}\sum_{l=0}^{\kappa-1}\sum_{p=0}^{\kappa-1} \sum_{\substack{j;r,s\\ j\in\frac{1}{2}\mathbb{Z}_+\\ r,s=-j,\dots,j}} e^{\frac{2\pi i}{\kappa}p(r+s-\nu)}\ e^{\frac{2\pi i}{\kappa}l(r-s)}\ \overline{D}^{j}_{r,s}(g_\alpha)\, D^{j}_{r,s}(g_\beta)\ \chi^{\rm Vir}_{j^2}(\tilde q) \tag{3.14}$$

Part of the exponentials can be absorbed into the D^j using the matrix

$$\Gamma_\kappa = \begin{pmatrix} e^{\frac{\pi i}{\kappa}} & 0\\ 0 & e^{-\frac{\pi i}{\kappa}}\end{pmatrix} \in \mathrm{SU}(2) \tag{3.15}$$

which satisfies

$$D^j_{r,s}(\Gamma_\kappa) = e^{\frac{2\pi i}{\kappa} r}\delta_{r,s} \tag{3.16}$$

and hence

$$e^{\frac{2\pi i}{\kappa} p(r+s)}\, D^j_{r,s}(g) = D^j_{r,s}(\Gamma^p_\kappa\, g\, \Gamma^p_\kappa)\ , \quad e^{\frac{2\pi i}{\kappa} p(r-s)}\, D^j_{r,s}(g) = D^j_{r,s}(\Gamma^p_\kappa\, g\, \Gamma^{-p}_\kappa)\ . \tag{3.17}$$

Now one can use the representation property of the D^j and perform the summation over r, s:

$$\mathcal{A}^{\mathrm{Vir}}_{\alpha\beta;\,\nu} = \frac{1}{\kappa^2}\sum_{l=0}^{\kappa-1}\sum_{p=0}^{\kappa-1}\sum_{j\in\frac{1}{2}\mathbb{Z}_+} e^{-\frac{2\pi i\nu\, p}{\kappa}}\ \mathrm{Tr}\, D^j\left(\Gamma^p_\kappa\, g^{-1}_\alpha\, \Gamma^{p+l}_\kappa\, g_\beta\, \Gamma^{-l}_\kappa\right)\ \chi^{\mathrm{Vir}}_{j^2}(\tilde{q}) \tag{3.18}$$

It remains to perform the modular transformation of the Virasoro characters, see e.g. [9] , and to combine the above expression with (3.12) for the parafermionic part. The overlap (3.7) becomes

$$\begin{aligned}\mathcal{A}_{\alpha\beta} &= \frac{\mathcal{N}^2}{2}\,\frac{1}{\kappa^2}\,\frac{2}{k} \sum_{\substack{(J',n')\\ \text{s.t. } 2J'+n' \text{ even}}}\ \sum_{\substack{\nu=-\kappa+1,\dots,\kappa\\ l=0,\dots,\kappa-1\\ p=0,\dots,\kappa-1}}\ \sum_{j\in\frac{1}{2}\mathbb{Z}_+} e^{\frac{2\pi i\nu}{2\kappa}(n'+n_\beta-n_\alpha-2p)} N^{J'}_{J_\alpha,J_\beta}\, \chi^{\mathrm{PF}}_{(J',n')}(q)\\ &\qquad\qquad \times\ \mathrm{Tr}\, D^j\left(\Gamma^p_\kappa\, g^{-1}_\alpha\, \Gamma^{p+l}_\kappa\, g_\beta\, \Gamma^{-l}_\kappa\right)\ \chi^{\mathrm{Vir}}_{j^2}(\tilde{q})\\ &= \frac{\sqrt{2}\,\mathcal{N}^2\, 2\kappa}{k^2} \sum_{\substack{(J',n')\\ \text{s.t. } 2J'+n' \text{ even}}}\ \sum_{\substack{l=0,\dots,\kappa-1\\ m\in\mathbb{Z}}} N^{J'}_{J_\alpha,J_\beta}\ \chi^{\mathrm{PF}}_{(J',n')}(q)\ \vartheta_{\frac{-\alpha_{Nl}(g_\alpha,g_\beta)}{\sqrt{2}\pi}+\sqrt{2}m}(q)\quad .\end{aligned} \tag{3.19}$$

To obtain the last line in (3.19), we have first performed the summation over ν, which fixes p to be $N := \frac{n'+n_\beta-n_\alpha}{2}$ – this is an integer due to the SU(2) fusion rules and the parafermion condition $2J+n$ even. Next, modular transformation of the combination of $\mathrm{Tr}\, D^j$ and Virasoro characters results (see e.g. [19,9]) in theta functions with highest weight given by the angles $\alpha_{Nl}(g_\alpha, g_\beta)$ defined in terms of the SU(2) trace

$$2\cos(\alpha_{Nl}(g_\alpha, g_\beta)) = \mathrm{Tr}_{\frac{1}{2}}\left(g^{-1}_\alpha\Gamma^N_\kappa\Gamma^l_\kappa\, g_\beta\,\Gamma^{-l}_\kappa\Gamma^N_\kappa\right)\ . \tag{3.20}$$

Since we are free to choose the overall normalisation factor in the definition (3.4) to be

$$\mathcal{N} = \left(\frac{\kappa}{\sqrt{2}}\right)^{3/2}\ ,$$

the result (3.19) shows that our boundary states for odd $k=\kappa^2$ do indeed satisfy Cardy's condition, which is the most important and usually most restrictive non-linear constraint to be imposed on conformal boundary conditions.

We now turn to the case when the *level is an even square*. As was pointed out at the end of Section 2, there exist additional parafermion Ishibashi states $|\frac{k}{4},\nu\kappa\rangle\rangle_{\rm tw}$ whenever k is even, namely for $J=\frac{k}{4}$, due to the field identification. This implies that the subspace (2.12) providing symmetry-breaking Ishibashi states needs be extended slightly and becomes

$$\begin{aligned}\bigoplus_{J=0,\dots,\frac{k}{2}} H_J^{SU(2)_k}\otimes\bar H_J^{SU(2)_k} \supset &\bigoplus_{\substack{J=0,\dots,\frac{k}{2}\\ \nu=-\kappa+1,\dots,\kappa\\ \text{s.t.}\,2J+\nu\,\text{even}\\ m_L,m_R\in\mathbb{Z}\\ l_L,l_R\in\mathbb{Z}_+}} H^{\rm PF}_{(J,\nu\kappa)}\otimes\bar H^{\rm PF}_{(J,-\nu\kappa)}\otimes H^{\rm Vir}_{|\,\kappa m_L+\frac{\nu}{2}|+l_L}\otimes\bar H^{\rm Vir}_{|\,\kappa m_R-\frac{\nu}{2}|+l_R}\\ &+\bigoplus_{\substack{\nu=-\kappa+1,\dots,\kappa\\ m_L,m_R\in\mathbb{Z}\\ l_L,l_R\in\mathbb{Z}_+}} H^{\rm PF}_{(\frac{k}{4},\nu\kappa)}\otimes\bar H^{\rm PF}_{(\frac{k}{4},-\nu\kappa+k)}\otimes H^{\rm Vir}_{|\,\kappa m_L+\frac{\nu}{2}|+l_L}\otimes\bar H^{\rm Vir}_{|\,\kappa(m_R-\frac12)-\frac{\nu}{2}|+l_R}.\end{aligned} \tag{3.21}$$

The additional Ishibashi states allow us to refine the boundary states (3.4), so as to resolve the field identification 'fixed point'. To this end, we make the ansatz (cf. [19])

$$\|\,g,\pm\,\rangle\rangle_{\rm tot}=\frac{1}{2}\left(\,\|\,g\,\rangle\rangle\pm\|\,g\,\rangle\rangle_{\rm tw}\right)\,, \tag{3.22}$$

where $\|\,g\,\rangle\rangle$ stands for the boundary state in (3.4) and $\|\,g\,\rangle\rangle_{\rm tw}$ is given by

$$\|\,g_\alpha;J_\alpha,n_\alpha\,\rangle\rangle_{\rm tw}=\ \mathcal{N}_{\rm tw}\sum_{\nu=-\kappa+1,\dots,\kappa}\ \sum_{\substack{j\in\frac12\mathbb{Z}_+\\ r+s=\kappa\rho+\nu+\frac{\kappa}{2}\\ r-s=\kappa\rho'-\frac{\kappa}{2}\\ \text{s.t.}\,\rho+\rho'\,\text{even}}} B^{{\rm PF}\ \alpha}_{(\frac{k}{4},\nu\kappa)}\,D^j_{r,s}(g_\alpha)\ |\tfrac{k}{4},\nu\kappa\rangle\rangle_{\rm tw}\otimes|j;r,s\rangle\rangle \tag{3.23}$$

with g_α and, for the time being, (J_α,n_α) as in $\|\,g_\alpha\rangle\rangle$. Compared to (3.4), the possible values of r and s summed over in (3.23) have been altered in response to the shift in the Virasoro label.

We need to verify that the overlaps of two boundary states of type $\|\,g,\pm\,\rangle\rangle_{\rm tot}$ still satisfy Cardy's conditions. First note that

$$\langle\langle\,g_\alpha\|\,q^{\frac12(L_0^{SU(2)_k}+\bar L_0^{SU(2)_k}-\frac{c}{12})}\,\|\,g_\beta\rangle\rangle_{\rm tw}=0$$

since the additional parafermionic Ishibashi states are orthogonal to the ones encountered for odd k. Thus the only new quantity we need to compute is

$$\mathcal{A}_{\alpha\beta;\,\rm tw}={}_{\rm tw}\langle\langle\,g_\alpha\|\,q^{\frac12(L_0^{SU(2)_k}+\bar L_0^{SU(2)_k}-\frac{c}{12})}\,\|\,g_\beta\rangle\rangle_{\rm tw}\ ,$$

which can be done along similar lines as for odd k, but with some differences: The offset by $\frac{\kappa}{2}$ in the r,s summation leads to an extra factor $(-1)^{l+N}$ from the Virasoro part. In

the parafermion contribution, there is no sum over J, instead $J=\frac{k}{4}$ is fixed and one has to exploit

$$S^{\mathrm{SU}(2)_k}_{\frac{k}{4},J}=\begin{cases}\sqrt{\frac{2}{k+2}}\,(-1)^J & J \text{ even}\\ 0 & J\text{ odd}\end{cases}\ .$$

This leads to an additional factor $(-1)^{J_\alpha+J_\beta+J'}=(-1)^N$ in the overlap – and it also implies that $\|\,g_\alpha;J_\alpha,n_\alpha\,\rangle\!\rangle_{\mathrm{tw}}$ is non-zero only for integer J_α, and that all integer J_α lead to the same $\|\,g_\alpha;J_\alpha,n_\alpha\,\rangle\!\rangle_{\mathrm{tw}}$. Altogether, one arrives at

$$\mathcal{A}_{\alpha\beta;\,\mathrm{tw}}=\mathcal{N}^2_{\mathrm{tw}}\,\frac{2}{k+2}\frac{1}{\kappa^2}\frac{2}{k}\sqrt{2}\sum_{\substack{(J',n')\\ J'\in\mathbb{Z},\ n'\mathrm{even}}}\ \sum_{\substack{l=0,\dots,\kappa-1\\ m\in\mathbb{Z}}}(-1)^l\ \chi^{\mathrm{PF}}_{(J',n')}(q)\,\vartheta_{\frac{-\alpha_{Nl}(g_\alpha,g_\beta)}{\sqrt{2}\pi}+\sqrt{2}m}(q) \tag{3.24}$$

where $N=\frac{n'+n_\beta-n_\alpha}{2}$ just as in (3.19). Choosing the normalisation $\mathcal{N}_{\mathrm{tw}}$ such that the prefactor of the sum disappears, we see that the sum of (3.24) and (3.19) is a sum of characters with positive integer coefficients if and only if we choose $J_\alpha=J_\beta=\frac{k}{4}$ in the boundary state $\|\,g_\alpha;J_\alpha,n_\alpha\,\rangle\!\rangle$; to see this, recall that $N^{J'}_{\frac{k}{4},\frac{k}{4}}=1$ precisely if J' is integer. Summarizing, we find two additional families of boundary states for even level $k=\kappa^2$,

$$\|\,g_\alpha;n_\alpha,n^{\mathrm{tw}}_\alpha;\pm\,\rangle\!\rangle_{\mathrm{tot}}=\frac{1}{2}\left(\,\|\,g_\alpha;\tfrac{k}{4},n_\alpha\,\rangle\!\rangle\pm\|\,g_\alpha;0,n^{\mathrm{tw}}_\alpha\,\rangle\!\rangle_{\mathrm{tw}}\right)\ .$$

We have thus constructed families of symmetry-breaking boundary states for $\mathrm{SU}(2)_k$ for any square level $k=\kappa^2$. They are parametrised by discrete labels J_α,n_α for the parafermionic degrees of freedom and SU(2) elements g_α for the Virasoro part. There are, however, identifications within those SU(2) families. To see this, note that the representation matrix elements $D^j_{r,s}(g)$ from (3.2) satisfy

$$D^j_{r,s}\big(\Gamma_\kappa\,g\,\Gamma_\kappa^{-1}\big)=e^{\frac{2\pi i}{\kappa}(r-s)}\ D^j_{r,s}(g)\ \ ,$$

and that they show up in the boundary state only for $r-s\equiv 0\ (\mathrm{mod}\,\kappa)$ for k odd and for $r-s\equiv\frac{\kappa}{2}\ (\mathrm{mod}\,\kappa)$ for k even. In the former case, the $\mathbb{Z}_\kappa$-action $g\mapsto\Gamma_\kappa g\Gamma_\kappa^{-1}$ leaves the boundary state invariant, in the latter case it swaps the $\|\,g,+\,\rangle\!\rangle_{\mathrm{tot}}$ with the $\|\,g,-\,\rangle\!\rangle_{\mathrm{tot}}$ branch. This means that instead of SU(2), the paramaters g_α in our boundary states (3.4) and (3.22) take values in

$$g_\alpha\in\mathrm{SU}(2)/\mathbb{Z}_\kappa\ \ .$$

Equivalently, we can restrict to the + sign in (3.22) and take $g_\alpha\in\mathrm{SU}(2)/\mathbb{Z}_{\frac{\kappa}{2}}$ for even $k=\kappa^2$.

4. Symmetry-breaking boundary states when k is not a square

We can also construct boundary states with a reduced PF $\times$ Vir symmetry when the SU(2) level k is not a square. Once more starting from the decomposition of the SU(2) bulk state space

$$\bigoplus_{J=0,\dots,\frac{k}{2}} H_J^{SU(2)_k} \otimes \bar{H}_J^{SU(2)_k} = \bigoplus_{\substack{J=0,\dots,\frac{k}{2} \\ n_l,n_r=-k+1,\dots,k \\ m_l,m_r\in\mathbb{Z} \\ \text{s.t.}\,2J+n_l\,\text{even} \\ \text{s.t.}\,2J+n_r\,\text{even}}} H_{(J,n_l)}^{\mathrm{PF}} \otimes \bar{H}_{(J,n_r)}^{\mathrm{PF}} \otimes H_{n_l}^{U(1)_k} \otimes \bar{H}_{n_r}^{U(1)_k} \quad (4.1)$$

from before, we again concentrate on those U(1) modules which break up into an infinite number of Virasoro irreducibles as in (2.9). We see from (2.6) that for non-square k this only happens in the vacuum module, $h_{(m_l,n_l)} = \bar{h}_{(m_r,n_r)} = 0$.
The subspace to which we associate symmetry-breaking Ishibashi states is thus

$$\bigoplus_{J=0,\dots,\frac{k}{2}} H_J^{SU(2)_k} \otimes \bar{H}_J^{SU(2)_k} \supset \bigoplus_{\substack{J=0,\dots,\frac{k}{2} \\ \text{s.t.}\,2J\,\text{even} \\ l_L,l_R\in\mathbb{Z}}} H_{(J,0)}^{\mathrm{PF}} \otimes \bar{H}_{(J,0)}^{\mathrm{PF}} \otimes H_{l_L}^{\mathrm{Vir}} \otimes \bar{H}_{l_R}^{\mathrm{Vir}} \,, \quad (4.2)$$

where we have used the explicit decomposition $H_0^{U(1)} = \bigoplus_{l=0}^{\infty} H_l^{\mathrm{Vir}}$ to rewrite the chiral U(1) modules in terms of Virasoro modules. For these values of h and $\bar{h}$, the SU(2) matrix elements $D^j_{r,s}(g)$ defined in (3.2) become the j^{th} Legendre polynomial $P_j(x)$, see [10,7] and also [9]. This follows from a simple rearrangement of (3.2) where now $r = s = 0$.

As a consequence, the boundary states (3.4) have to be altered slightly; we define

$$\| \, x; J_\alpha \rangle\!\rangle = \mathcal{N} \sum_{\substack{J=0,\dots,\frac{k}{2} \\ \text{s.t. } 2J \text{ even}}} \sum_{l\in\mathbb{Z}_+} B_{(J,0)}^{\mathrm{PF}\ (J_\alpha,0)} \, P_l(x) \, |J,0\rangle\!\rangle \otimes |l\rangle\!\rangle \qquad \text{for } x \in [-1,1] \,. \quad (4.3)$$

Computing the overlap of two such boundary states, and thus verifying that they satisfy Cardy's condition, is easier in this case since parafermionic and Virasoro parts decouple. Moreover, the field identification (2.1) plays no role in evaluating the parafermionic contribution since there is no Ishibashi state associated to the state space (4.2) with $|J,0\rangle\!\rangle = |\frac{k}{2} - J, k\rangle\!\rangle$. We find that

$$\begin{aligned} \mathcal{A} &= \langle\!\langle x_\alpha; J_\alpha \| \, \tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})} \, \| \, x_\beta; J_\beta \rangle\!\rangle \\ &= \mathcal{N}^2 \sum_{\substack{J=0,\dots,\frac{k}{2} \\ \text{s.t. } 2J \text{ even}}} \sum_{l\in\mathbb{Z}_+} \overline{B}_{(J,0)}^{\mathrm{PF}\ \alpha} \, B_{(J,0)}^{\mathrm{PF}\ \beta} \, P_l(x_\alpha) P_l(x_\beta) \, \chi_{(J,0)}^{\mathrm{PF}}(\tilde{q}) \, \chi_{l^2}^{\mathrm{Vir}}(\tilde{q}) \\ &= \frac{2\mathcal{N}^2}{k} \sum_{\substack{(J',n') \\ \text{s.t. } 2J'+n' \text{ even}}} N_{J_\alpha,J_\beta}^{J'} \, \chi_{(J',n')}^{\mathrm{PF}}(q) \sum_{l\in\mathbb{Z}_+} P_l(x_\alpha) P_l(x_\beta) \, \chi_{l^2}^{\mathrm{Vir}}(\tilde{q}) \,. \end{aligned} \quad (4.4)$$

The computation of the Virasoro contribution, involving a modular transformation of the Virasoro characters, was presented in [10]:

$$\sum_{l\in\mathbb{Z}_+} P_l(x_\alpha)P_l(x_\beta)\,\chi^{\rm Vir}_{l^2}(\tilde q) = \frac{1}{\sqrt{2}\pi^2}\int_0^\pi d\phi'\int_0^\pi d\phi\,\sum_{n\in\mathbb{Z}}\vartheta_{\frac{1}{\sqrt{2}}(n+\frac{t}{2\pi})}(q) \tag{4.5}$$

where t is defined through (using $x_\alpha =: \cos\theta_\alpha$ etc.)

$$\cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\phi}{2}\ , \quad \cos\theta = \cos\theta_\alpha\cos\theta_\beta - \sin\theta_\alpha\sin\theta_\beta\cos\phi'\ . \tag{4.6}$$

The final result for the overlap between two symmetry-breaking boundary states $\|x_\alpha; J_\alpha\rangle\rangle$ is therefore a continuous band spectrum

$$\mathcal{A} = \frac{2\mathcal{N}^2}{k\sqrt{2}\pi^2}\int_0^\pi d\phi_2\int_0^\pi d\phi \sum_{\substack{(J',n')\\ \text{s.t. } 2J'+n'\text{ even}}}\ \sum_{n\in\mathbb{Z}} N^{J'}_{J_\alpha,J_\beta}\,\chi^{\rm PF}_{(J',n')}(\tilde q)\,\chi^{\rm Vir}_{\frac14(n+\frac{t}{2\pi})^2}(\tilde q)\ . \tag{4.7}$$

5. Comparison with maximally symmetric SU(2)$_k$ boundary states

In this section, we would like to investigate the relation between our symmetry-breaking boundary states and others constructed previously in the literature, in particular maximally symmetric boundary conditions for the SU(2) WZW model. The latter are given by ordinary Cardy states and by rotated Cardy branes, which can be written as (see e.g. [13])

$$\|\,J_\alpha\rangle\rangle_{\lambda_a} = \exp\{i\,\lambda_a\,J_0^a\}\ \|\,J_\alpha\rangle\rangle_0 \tag{5.1}$$

where $\|\,J_\alpha\rangle\rangle_0$ is a Cardy boundary state and λ_a, $a = 1, 2, 3$, are rotation parameters. These states preserve a full SU(2)$_k$ symmetry (albeit a twisted one) due to the rotated gluing conditions

$$\big(\mathrm{Ad}_h(J^b_m) + \bar J^b_{-m}\big)\ \|\,J_\alpha\rangle\rangle_{\lambda_a} = 0$$

involving the adjoint action of the group element $h := \exp\{\lambda_a t^a\}$ on the currents J^b_m. Semi-classically, these branes correspond to conjugacy classes in the WZW target [4].

Since the affine Lie algebra preserved by $\|\,J_\alpha\rangle\rangle_{\lambda_a}$ is rotated relative to the one preserved by Cardy boundary states, the overlap of two branes with non-zero relative angle no longer decomposes into full affine Lie algebra characters, but is close to the overlaps we found in Section 3 for level $k = \kappa^2$. (We will restrict to square level in the following and assume that k is odd for convenience.)

We will now show that, for the choice $g_\alpha = \mathbf{1}_2$ and $n_\alpha = 0$, our boundary states $\|\,g_\alpha; J_\alpha, n_\alpha\rangle\rangle$ from eq. (3.4) coincide with superpositions of rotated Cardy branes (with

Cardy label J_α). Let us specialise to $\lambda_a = \lambda\,\delta_{a,3}$ in (5.1) and consider the action of the rotation on the $\mathrm{SU}(2)_k$ Ishibashi states, which can be decomposed into parafermion and U(1) Ishibashi states:

$$\begin{aligned}\exp\{i\,\lambda J_0^3\}\,|J\rangle\!\rangle^{\mathrm{SU}(2)} &= \sum_{n,m;2J+n \text{ even}} |J,n\rangle\!\rangle^{\mathrm{PF}} \otimes\ \exp\{i\,\lambda J_0^3\}\,|n,m\rangle\!\rangle^{\mathrm{U}(1)} \\ &= \sum_{n,m;2J+n \text{ even}} |J,n\rangle\!\rangle^{\mathrm{PF}} \otimes\ e^{i\,\lambda\,q_{n,m}}\,|n,m\rangle\!\rangle^{\mathrm{U}(1)}\end{aligned} \tag{5.2}$$

(Note that no phases or unitary transformations occur in the decomposition of $|J\rangle\!\rangle$ into $\mathrm{PF}\times\mathrm{Vir}$ Ishibashi states, since an Ishibashi state $|i\rangle\!\rangle$ can be regarded as a projector in $\mathrm{End}\mathcal{H}_i$), as follows from [29] and the arguments presented in [30].) We have used that the U(1) current J_0^3 commutes with the parafermion sector and is diagonal on each U(1) irrep, producing the charge $q_{n,m} = 2\,k\,(m + \frac{n}{2k})$.
Now let us sum over a discrete set of rotation angles $\lambda_l = \frac{2\pi\,l}{\kappa}$ for $l = 0,1,\ldots,\kappa-1$. Due to the U(1) charges present in the $\mathrm{SU}(2)_k$ theory, the l-summation projects out those U(1) Ishibashi states that do not satisfy $n = \nu\,\kappa$ and leaves only those that yield degenerate Virasoro representations – which were precisely the ones we restricted to in our ansatz for symmetry-breaking boundary states (3.4). Furthermore, it is straightforward to see that the superposition of κ rotated Cardy branes $\|\,J_\alpha\rangle\!\rangle_{\lambda_l}$ coincides with the symmetry-breaking boundary state $\|\,g_\alpha = \mathbf{1}_2; J_\alpha, n_\alpha = 0\rangle\!\rangle$ up to overall normalisation. That the latter agrees, as well, can be seen by counting the number of identity operators in the partition function of the superposition and in self-overlap (3.19) computed in Section 3. *

For generic g_α, the boundary states (3.4) cannot be written as superpositions of rotated Cardy branes, but for another special choice of the g_α parameters, namely

$$g_\alpha = g_N := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ ,$$

they resemble the 'B-type' boundary states constructed in [19], see also [32,20]: One can easily show that

$$\|\,g_N; J_\alpha, n_\alpha\rangle\!\rangle \sim \sum_{J;\ 2J \text{ even}} B^{\mathrm{PF}\ (J,0)}_{(J_\alpha,n_\alpha)}\ |J,0\rangle\!\rangle^{\mathrm{PF}} \otimes \|N\rangle\!\rangle \tag{5.3}$$

* Note that the decomposition (5.2) also allows to show that the overlap between $\|\,g_N; J_\alpha, n_\alpha\rangle\!\rangle$ and an $\mathrm{SU}(2)_k$ Cardy boundary state yields a good partition function. One needs to observe that, from the latter, only those $\mathrm{PF}\times\mathrm{U}(1)$ Ishibashi states which have $\lambda_l\,q_{m,n} \in 2\pi\mathbb{Z}$ contribute to the overlap.

where the last factor is a superposition of κ free boson Neumann boundary states with evenly spaced Wilson lines, see [8,9]. In contrast to the B-type states from [19], the boundary states (5.3) only respect $\mathrm{PF}\times\mathrm{U}(1)$ symmetry, not $\mathrm{PF}\times\mathrm{U}(1)_k$. However, the arguments from [19] supporting the semi-classical interpretation of B-type branes as three-dimensional objects seem applicable to (5.3), as well.

Taking this for granted, we are led to conclude that our families of symmetry-breaking boundary states interpolate between superpositions of (generically 2-dimensional) rotated Cardy branes (for $g_\alpha = \mathbf{1}_2$) and "3-dimensional" branes for $g_\alpha = g_N$. At both these points, a $\mathrm{PF}\times\mathrm{U}(1)$ symmetry is preserved, while the symmetry is broken down to $\mathrm{PF}\times\mathrm{Vir}$ for generic g_α, and no-where enhanced to full $\mathrm{SU}(2)_k$.

Note that the 'Legendre boundary states' (4.3) for non-integer $\sqrt{k}$ show a similar behaviour at the endpoints $x = \pm 1$ of the continous parameter: For $x = 1$, the Virasoro contributation can be written as an integral over ordinary Dirichlet boundary states, while $x = -1$ corresponds to an integral of Neumann boundary states over the dual circle, see [10].

6. Extension to other group targets

Building on the ideas of [19] and [20], it is rather straightforward to generate symmetry-breaking boundary states for higher rank WZW models from the ones for SU(2) presented above. SU(2) can be embedded into any compact Lie group, and we can use cosets of the form

$$G = G/SU(2) \ \times \ SU(2)$$

to break the underlying G_k symmetry. One can, e.g., build boundary states of the type

$$\| \, (\rho_\alpha, J_\alpha)\rangle\!\rangle = \sum_{(\mu,J)} B^{(\rho_\alpha,J_\alpha)}_{(\mu,J)} \ |\mu, J\rangle\!\rangle \otimes |J\rangle\!\rangle \tag{6.1}$$

where $|\mu, J\rangle\!\rangle$ and $|J\rangle\!\rangle$ in (6.1) are Ishibashi states of the coset $G/SU(2)_k$ and $SU(2)_k$ theories respectively; the possible (μ, J) in the summation may be subject to non-trivial field identification and branching selection rules, as seen before in the PF theory. The coefficients $B^{(\rho_\alpha,J_\alpha)}_{(\mu,J)}$ have to be chosen such that Cardy's condition is satisfied, and one possibility was presented in [20], namely

$$B^{(\rho_\alpha,J_\alpha)}_{(\mu,J)} = \frac{S^G_{\rho_\alpha\mu}}{\sqrt{S^G_{0\mu}}} \ \frac{\overline{S}^{\mathrm{SU}(2)_k}_{J_\alpha J}}{\overline{S}^{\mathrm{SU}(2)_k}_{0J}} \ . \tag{6.2}$$

This uses the modular S-matrices of the G and SU(2) WZW theories, and in general leads to a spectrum without G-symmetry even though trivial gluing conditions apply in the coset and in the SU(2) factor.

We can reduce the symmetry even further by incorporating our boundary states (3.4) into the decomposition (6.1), in place of the SU(2) Ishibashi states. We simply modify (6.1) by decomposing the latter Ishibashi states into ones for PF $\times$ Vir as before and propose symmetry-breaking boundary states for the G WZW model of the form

$$\| \rho_\alpha, J_\alpha, n_\alpha g_a \rangle\rangle = \mathcal{N} \sum_{(\mu,J)} \sum_{\nu=-\kappa+1}^{\kappa} \sum_{\substack{j\in\frac{1}{2}\mathbb{Z}_+ \\ r+s=\kappa\rho+\nu \\ r-s=\kappa\rho' \\ \text{s.t.}\,\rho+\rho'\,\text{even}}} B^{\alpha}_{(\mu,J,\nu\kappa;j,r,s)} \, |\mu, J\rangle\rangle \otimes |J, \nu\kappa\rangle\rangle \otimes |j; r, s\rangle\rangle \ ; \quad (6.3)$$

here we assume that the level $k = \kappa^2$ is a square, and the coefficients $B^{\alpha}_{(\mu,J,\nu\kappa;j,r,s)}$ are given by

$$B^{\alpha}_{(\mu,J,\nu\kappa;j,r,s)} = \frac{S^G_{\rho_\alpha \mu}}{\sqrt{S^G_{0\mu}}} \, \frac{\overline{S}^{PF}_{(J_\alpha,n_\alpha),(J,\nu\kappa)}}{\overline{S}^{PF}_{(0,0),(J,\nu\kappa)}} \, D^j_{r,s}(g) \ . \quad (6.4)$$

For convenience, we assume that the permissible representations of the $G/\mathrm{SU}(2)$ coset theory are not subject to any field identifications and hence there are no branching selection rules, unlike the PF theory. Although this might restrict the possible coset models at our disposal, there are still many models (e.g. the SU(3)/SU(2) theory) for which this assumption holds and hence for which our construction will still be valid. This assumption will simplify the resulting calculation since we can split the sum over (μ, J) in (6.3) into two sums; one running over the possible irreducible representations of G, the other over those of $\mathrm{SU}(2)_\mathrm{k}$. The remaining constraint $2J + \nu\kappa \equiv 0 \pmod 2$ can be dealt with as before – which also suggests that our simplifying assumption on the G/SU coset can be relaxed.

In essence, the computations required to verify Cardy's constraint proceed very much along the lines of Section 3. The only additional pieces of information needed to calculate the overlap are:

(1) The S-matrix $S^{G/\mathrm{SU}(2)_k}_{(\mu,J),(\mu'',J'')}$ for modular transformation of coset characters may be decomposed into the S matrices of the two independent theories through the formula (see e.g. [33,20])

$$S^{G/\mathrm{SU}(2)_k}_{(\mu,J),(\mu'',J'')} = S^G_{\mu,\mu''} \, \overline{S}^{\mathrm{SU}(2)_\mathrm{k}}_{J,J''} \ . \quad (6.5)$$

(2) The ratios $S^{\mathrm{SU}(2)_\mathrm{k}}_{J_\alpha,J} / S^{\mathrm{SU}(2)_k}_{0,J}$ of S-matrix elements (the generalised quantum dimensions) form a representation of the fusion algebra, i.e.

$$\frac{S^{\mathrm{SU}(2)_k}_{J_\alpha,J}}{S^{\mathrm{SU}(2)_k}_{0,J}} \, \frac{S^{\mathrm{SU}(2)_k}_{J_\beta,J}}{S^{\mathrm{SU}(2)_k}_{0,J}} = \sum_{J''} N^{J''}_{J_\alpha,J_\beta} \, \frac{S^{\mathrm{SU}(2)_k}_{J'',J}}{S^{\mathrm{SU}(2)_k}_{0,J}}. \quad (6.6)$$

The overlap of two boundary states as in (6.3), properly normalised, is then given by

$$
\begin{aligned}
&\langle\!\langle\, \alpha \| \, q^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})} \, \| \, \beta \rangle\!\rangle && (6.7)\\
&\quad = \sum_{\substack{(\mu,J),\ J'',\ (J',n')\\ l=0,\ldots,\kappa-1,\ m\in\mathbb{Z}}} N^{G\,\mu}_{\rho_\alpha,\rho_\beta} N^{J''}_{J_\alpha,J_\beta} N^{J}_{J',J''}\ \chi^{G/\mathrm{SU}(2)_k}_{(\mu,J)}(q)\, \chi^{\mathrm{PF}}_{(J',n')}(q)\, \vartheta_{\frac{-\alpha_{Nl}(g_\alpha,g_\beta)}{\sqrt{2}\pi}+\sqrt{2}m}(q) \ .
\end{aligned}
$$

It is easy to generalize this construction to produce other symmetry-breaking boundary states for the Lie group $\mathrm{SU}(M)$ by realising it as a string of cosets of the form

$$
\begin{aligned}
\mathrm{SU}(M) \cong \big(\mathrm{SU}(M)/\mathrm{SU}(M-1)\big) \times \big(\mathrm{SU}(M-1)/\mathrm{SU}(M-2)\big) \times \\
\times \cdots \times \big(\mathrm{SU}(3)/\mathrm{SU}(2)\big) \times \mathrm{SU}(2) \ .
\end{aligned} \tag{6.8}
$$

Again, one has to be aware of the possibililty of field identification and branching selection rules appearing in the corresponding decomposition of the WZW state space; we can avoid this problem, however, if we demand that the generators of $\mathrm{SU}(l-1)$ form the upper-left block in $\mathrm{SU}(l)$.

The sequence of embeddings can be exploited to produce symmetry-breaking boundary states, as discussed in [34]. We can again generalise the construction there by using one of our symmetry-breaking boundary states for the last SU(2) factor. We arrive at

$$
\|\alpha\rangle\!\rangle = \sum_{\substack{(\rho_M,\rho_{M-1}),\ (\rho_{M-1},\rho_{M-2}),\cdots\\ (J,\nu\kappa),\ j;r,s}} B^{\alpha}_{(\rho_M,\cdots,j;r,s)}\ |\rho_M,\rho_{M-1}\rangle\!\rangle \otimes \cdots \otimes |J,\nu\kappa\rangle\!\rangle \otimes |j;r,s\rangle\!\rangle \tag{6.9}
$$

where (ρ_l,ρ_{l-1}) denote the irreducible representations of the coset $\mathrm{SU}(l)/\mathrm{SU}(l-1)$, and where the coefficients are

$$
B = \frac{S^{\mathrm{SU}(M)}_{(\rho_M)_\alpha\rho_M}}{\sqrt{S^{\mathrm{SU}(M)}_{0,\rho}}}\, \frac{\overline{S}^{\mathrm{SU}(M-1)}_{(\rho_{M-1})_\alpha,\rho_{M-1}}}{\overline{S}^{\mathrm{SU}(M-1)}_{0,\rho_{M-1}}}\, \frac{\overline{S}^{\mathrm{SU}(M-2)}_{(\rho_{M-2})_\alpha,\rho_{M-2}}}{\overline{S}^{\mathrm{SU}(M-2)}_{0,\rho_{M-2}}} \cdots \frac{\overline{S}^{\mathrm{PF}}_{(J,\nu\kappa),(J_\alpha,n_\alpha)}}{\overline{S}^{\mathrm{PF}}_{(0,0),(J,\nu\kappa)}}\ D^{j}_{r,s}(g) \ . \tag{6.10}
$$

Computation of the overlap of two such boundary states leads to

$$
\begin{aligned}
Z_{\alpha\beta}(q) = \sum_{\substack{(\rho'_M,\rho'_{M-1}),\ (\rho''_{M-1},\rho'_{M-2})\cdots\\ (J'',n')\\ l=0,\cdots,\kappa-1;\ m\in\mathbb{Z}}}\ \sum_{\substack{\rho'''_{M-1},\ \rho'''_{M-2}\cdots\\ J'''}} N^{\rho'_M}_{(\rho_M)_\alpha(\rho_M)_\beta} \prod_{l=2}^{M-1} N^{\rho'''_l}_{(\rho_l)_\alpha(\rho_l)_\beta} N^{\rho'''_l}_{\rho'_l\rho''_l} \\
\times\, \chi^{\frac{M}{M-1}}_{(\rho'_M,\rho'_{M-1})}(q)\chi^{\frac{M-1}{M-2}}_{(\rho''_{M-1},\rho'_{M-2})}(q)\cdots\chi^{\mathrm{PF}}_{(J'',n')}(q)\ \vartheta_{\frac{-\alpha_{Nl}(g_\alpha,g_\beta)}{\sqrt{2}\pi}+\sqrt{2}m}(q)
\end{aligned}
$$

where $\chi^{\frac{M-1}{M-2}}_{(\mu'',\nu')}(\tilde{q})$ are characters of the coset $\mathrm{SU}(M-1)/\mathrm{SU}(M-2)$.

7. Conclusions and open questions

We have presented a construction of boundary states for SU(2) WZW models that break the symmetry to PF $\times$ Vir, using a coset decomposition of SU(2) and the most general conformal boundary states for a free boson. The moduli space of those boundary states depends on whether the level k of the SU(2) theory is a square or not: in the latter case, we find families parametrised by $x \in [-1, 1]$, for $k = \kappa^2$ we find a discrete quotient of SU(2), namely $\mathrm{SU}(2)/\mathbb{Z}_{\frac{\kappa}{2}}$ for even k and $\mathrm{SU}(2)/\mathbb{Z}_\kappa$ for odd k. At special points of those 3-dimensional families, our boundary states are superpositions of κ Cardy branes at relative angles, for generic parameters they are elementary branes – and so far lack a target space interpretation (which might be identified using the methods of [35,19]).

We have performed the most important consistency check and shown that the new symmetry-breaking boundary states satisfy Cardy's condition, but one obvious open question is to check other sewing relations like the cluster condition. For the Legendre boundary states from Section 4, this follows directly from the analysis given in [10], but in the case of $k = \kappa^2$ one needs to deal with non-trivial field identification issues.

It would be interesting to see whether the three-dimensional family parametrised by g_α in (3.4) is one of marginal boundary deformations. Already the counting of boundary operators with conformal dimension one that are supported by these boundary states is involved, since the spectrum displays a complicated g_α-dependence due to (3.20). However, one can check that there are always at least three marginal fields in the spectrum (3.19): the parafermion vacuum (labelled $J' = n' = 0$) tensored with dimension one states counted by the theta functions $\vartheta_0(q)$ and $\vartheta_{\pm\sqrt{2}}(q)$. Whether the associated operators are truly marginal and responsible for the three-dimensional family of symmetry-breaking boundary states is, of course, much more difficult to decide.

There are somewhat related questions concerning renormalisation group flows: All maximally symmetric branes in WZW models (Cardy branes as well as rotated ones) can be viewed as condensates of D0-branes – in particular they arise as (perturbative) RG fixed points, or as solutions to the equations of motion of the effective action computed in [36], see also [37,38]. It is known, at least for higher rank groups, that also symmetry-breaking branes can arise from such condensation processes [39]– in fact, the ground state of the effective action does not preserve the maximal symmetry – but it is unclear whether the boundary states constructed in this paper have a (perturbative) description for large k.

On the other hand, since their construction starts from a brute force breaking of the symmetry to PF $\times$ U(1), it should be relatively straightforward to study their behaviour under marginal bulk deformation of the $\mathrm{SU}(2)_k$ theory under $J^3(z)J^3(\bar{z})$, see e.g. [40,41]. In view of the recent results of [42] on bulk deformations of conformal U(1) boundary states, one would not expect our symmetry-breaking SU(2) boundary states to decay into other boundary conditions under such marginal bulk deformations.

While we have given simple generalisations to higher rank group targets exploiting embeddings of SU(2), one may conjecture that there exist other boundary states constructed via a decomposition of the G symmetry in which the Virasoro algebra from the SU(2) case is extended to a higher Casimir W-algebra, e.g. $W(2,3,\ldots,N)$ for SU(N). It is also tempting to suggest that for higher rank groups G, one can construct symmetry-breaking boundary states which involve representation matrix elements of G in place of the $D^j_{r,s}(g)$ for $g \in \mathrm{SU}(2)$. As a first step to verify this, one should analyse the G WZW model for level $k = 1$ in detail.

What may be more relevant for applications to string theory is a generalisation to supersymmetric WZW models on the one hand (in particular to supersymmetric SU(2), which shows up in the world-sheet description of NS 5-branes [23,43]), and to coset models on the other, which would open up the possibility of constructing new families of boundary states in CFTs which are important for string model building.

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